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1979 J. Phys. A: Math. Gen. 12 75

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Percolation with restricted valence

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Received 3 July 1978

Abstract. We have enumerated exactly the square lattice site animals with vertex degree less than or equal to three and with up to fourteen sites in the cluster. Using series analysis techniques estimates have been made of the location, exponent and amplitude of the principal singularity in the animal generating function for maximum vertex degree $v = 2, 3$ and 4. We have shown rigorously that the dominant singularity in the generating function must have a different exponent for $v = 2$ than for $v = 3$ and the numerical results suggest that the generating function diverges with an exponent of $\frac{4}{3}$ for $v = 2$ but logarithmically for both $v = 3$ and $v = 4$.

1. Introduction

One of the areas of lattice statistics which is receiving considerable attention is the percolation problem (for a review, see Essam 1972). In this paper we consider a particular aspect of this problem, the number of lattice animals, i.e. the number of connected clusters of n sites. Several important rigorous results are available for this problem; for instance Klarner (1967) showed that the animals are super-multiplicative which implies that, if a_n is the number of site animals with n sites,

$$\sup_{n>0} n^{-1} \ln a_n = \lim_{n \rightarrow \infty} n^{-1} \ln a_n \equiv \ln \lambda \quad (1.1)$$

and several workers have derived rigorous bounds for λ (e.g. Klarner and Rivest 1973, Whittington and Gaunt 1978).

To obtain more detailed information on the asymptotic behaviour of a_n , the lattice animals have been enumerated exactly for small n and, assuming an asymptotic form such as

$$a_n \sim n^{-\tau} \lambda^n \quad (1.2)$$

estimates have been made of τ and λ (Sykes and Glen 1976, Sykes *et al* 1976, Gaunt *et al* 1976, Guttmann and Gaunt 1978).

In the usual percolation problems, percolation can occur from a site to any neighbouring lattice site, with given probability which is independent of the condition of other neighbouring sites. A generalisation of this problem is to consider *restricted percolation* in which the valence (or vertex degree) of any site cannot exceed some prescribed maximum value. For instance, for the square lattice, restricting the valence

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to a maximum value of three would rule out all clusters with a vertex of degree four. One reason for our interest in this generalisation is that it allows contact to be made with the excluded volume problem. If the maximum allowed valence (v) is set equal to two, the resulting clusters are the neighbour-avoiding walks and neighbour-avoiding polygons. (Notice that they are neighbour-avoiding because site animals are strong embeddings, i.e. section graphs of the lattice. Bond animals would correspond to self-avoiding walks and self-avoiding polygons.)

In this paper we shall consider the square lattice site animals with $v = 2, 3$ and 4 , and we write the number of site animals with maximum v , having n sites in the cluster as $a_n(v)$. The values of $a_n(4)$ up to $n = 19$ are available in Sykes and Glen (1976), and Domb, Gillis and Wilmers (1963, unpublished) and Hioe (1967) have enumerated the neighbour-avoiding walks on the square lattice with $n \leq 19$, which form the dominant contribution to $a_n(2)$. For $a_n(3)$ we have carried out exact enumerations for $n \leq 14$. In § 2 we make some comments on the super-multiplicative property and its consequences while § 3 contains our analysis of the exact enumeration data.

2. Asymptotic behaviour of site animals with restricted valence

To prove the existence of the connective constant $\lambda(3)$ (defined below) for the square lattice with restricted valence 3, the only point of interest is whether the 'usual' joining operation used in deriving the super-multiplicative property will violate the restricted valence condition.

We consider a particular cluster with m sites and define the top (bottom) site in the cluster as the left-most (right-most) site in the top (bottom) row of sites. Let the coordinates of these sites be (x_T, y_T) and (x_B, y_B) for one particular m -animal and (x'_T, y'_T) and (x'_B, y'_B) for a particular n -animal. Now translate so that $x'_T = x_B$ and $y'_T = y_B - 1$ and add a bond joining the bottom site of one to the top site of the other animal. The resulting graph is clearly a site animal with $(m + n)$ sites. Moreover, in the original m -animal the bottom site had a valence of not more than 2. (If the valence were 3 or 4 there must have been a site with coordinates $(x_B, y_B - 1)$ or $(x_B + 1, y_B)$, or both, which violates the definition of (x_B, y_B) as the bottom site.) A similar argument applies for the top site of the n -animal. Consequently the maximum valence of any site in the $(m + n)$ -animal is 3. Hence

$$a_m(3)a_n(3) \leq a_{m+n}(3) \quad (2.1)$$

and, by the usual arguments

$$0 < \sup_{n>0} n^{-1} \ln a_n(3) = \lim_{n \rightarrow \infty} n^{-1} \ln a_n(3) \equiv \ln \lambda(3) < \infty. \quad (2.2)$$

Using $a_{14}(3) = 5\,512\,502$ (see table 1) gives $\lambda(3) \geq 3.0305 \dots$. An improved lower bound, namely

$$\ln \lambda(3) \geq m^{-1} \ln(2a_m(3)) \quad (2.3)$$

may be obtained using an argument exactly analogous to that employed by Whittington and Gaunt (1978). This yields $\lambda(3) \geq 3.1843 \dots$ which is still well below our best numerical estimate of 3.950 ± 0.005 (see table 3) but greater than 3 which is a rigorous upper bound on $\lambda(2)$.

Notice that the construction leading to (2.1) fails for $v = 2$. In this case, the number of site animals is given by

$$a_n(2) = [n-1]_c + [n]_0 \quad (2.4)$$

where $[n]_c$ is the number of n -edge strongly embedded simple chains, i.e. the number of undirected neighbour-avoiding n -step walks, and $[n]_0$ is the number of (undirected, unrooted) n -edge strongly embedded polygons. It is trivial to prove that

$$\lim_{n \rightarrow \infty} n^{-1} \ln [n]_c = \inf_{n > 0} n^{-1} \ln [n]_c \geq \lim_{n \rightarrow \infty} n^{-1} \ln [n]_0 = \sup_{n > 0} n^{-1} \ln [n]_0, \quad (2.5)$$

so that (2.4) is dominated by the walk term. If we assume that we can write

$$a_n(v) \sim n^{-\tau(v)} \lambda(v)^n \quad (2.6)$$

then the above arguments show that $\tau(2) < 0$ while $\tau(3) > 0$. Hence a change in the exponent must occur from $v = 2$ to $v = 3$.

A corresponding argument for the d -dimensional hypercubic lattice is less conclusive. Extending the definition of top and bottom sites in the natural way leads to sites having valence not greater than d so $a_n(v)$ is super-multiplicative at least for $v \geq d + 1$, i.e. a change in exponent must occur somewhere between $v = 2$ and $v = d + 1$.

3. Series analysis

The values obtained by us for $a_n(3)$ are given in table 1 together with the corresponding results for $v = 2$ and 4. The values for $v = 4$ are reproduced from Sykes and Glen (1976) while the values for $v = 2$ have been obtained from data for neighbour-avoiding walks (Domb, Gillis and Wilmers, private communication, Hioe 1967).

Table 1. Numbers of clusters with n sites, having maximum valences 2, 3 and 4, strongly embeddable in the square lattice.

n	$a_n(2)$	$a_n(3)$	$a_n(4)$
1	1	1	1
2	2	2	2
3	6	6	6
4	15	19	19
5	34	62	63
6	82	208	216
7	198	712	760
8	471	2 481	2 725
9	1 122	8 764	9 910
10	2 664	31 308	36 446
11	6 334	112 872	135 268
12	14 979	410 046	505 861
13	35 506	1 499 290	1 903 890
14	83 770	5 512 502	7 204 874
15	198 086		27 394 666
16	466 468		104 592 937
17	1 100 818		400 795 844
18	2 588 302		1 540 820 542
19	6 097 830		5 940 738 676

The series analysis followed standard lines (Gaunt and Guttmann 1974) and we give only a brief account here. For $v = 2$, there are good theoretical reasons (Watson 1970) for believing that the exponent for neighbour-avoiding walks will be the same as for self-avoiding walks, and hence we expect $\tau(2) = -\frac{1}{3}$. The data reported here are in accord with this value, though with fairly large uncertainty. Assuming this value for the exponent, a Padé analysis of the data suggests $\lambda(2) = 2.316 \pm 0.002$. This is in precise agreement with the value reported by Hioe (1967). There is strong evidence of a second singularity for $v = 2$ on the circle of convergence of the generating function

$$G(x, v) = 1 + \sum_{n \geq 1} a_n(v) x^n \quad (3.1)$$

at $x = -1/\lambda(2)$, as is also found for self-avoiding walks (Watts 1975, Guttmann and Whittington 1978) and for neighbour-avoiding walks (Gaunt, unpublished, Hioe 1967). From (2.6) it follows that, close to $x = 1/\lambda(v)$,

$$G(x, v) \sim [A/(1-\tau)][(1-\lambda x)^{-1+\tau} - 1] \quad (3.2)$$

and for $v = 2$ Padé and ratio methods indicate $A = 0.305 \pm 0.004$.

For $v = 3$ we present the results of a ratio analysis in table 2. Defining $\lambda_n = a_n/a_{n-1}$ we form the linear extrapolants $\lambda'_n = n\lambda_n - (n-1)\lambda_{n-1}$ which suggest $\lambda > 3.94$. To estimate τ we form the sequence $n[1 - (\lambda_n/\lambda'_n)]$ which appears to be converging to $\tau = 1$. Letting τ go to unity in (3.2) we obtain

$$G(x, v) \sim -A \ln(1 - \lambda x) \quad (3.3)$$

Table 2. Ratio analysis for $v = 3$.

n	$\lambda'_n = n\lambda_n - (n-1)\lambda_{n-1}$	$n[1 - (\lambda_n/\lambda'_n)]$	$n\lambda_n/(n-1)$
7	3.832506	0.747815	3.993590
8	3.914866	0.879347	3.982343
9	3.915615	0.880708	3.974002
10	3.931395	0.913297	3.969268
11	3.933926	0.919144	3.965734
12	3.936749	0.926373	3.963099
13	3.939042	0.932818	3.961094
14	3.941253	0.939589	3.959568

so that the singularity is logarithmic. Hence we form the sequence $n\lambda_n/(n-1)$ which suggests $\lambda < 3.96$. As our best estimate we take $\lambda(3) = 3.950 \pm 0.005$. These results for τ and λ are completely consistent with a Padé analysis. From Padé and ratio analyses we estimate the amplitude A in (3.3) to be 0.355 ± 0.010 . Unlike the $v = 2$ case there is no sign of odd-even alternation in the ratio estimates and the poles of the Dlog Padé approximants show no evidence of a singularity at $-1/\lambda(3)$. The closest competing singularities appear to be a conjugate pair two to three times as far from the origin as the dominant physical singularity.

The series for $v = 4$ has already been discussed several times in the literature (e.g. Sykes and Glen 1976, Guttmann and Gaunt 1978). We summarise our final estimates in table 3 and merely comment that they are in excellent accord with previous work. Again the singularity appears to be logarithmic and there are no competing singularities close to the circle of convergence.

Table 3. Estimates of critical parameters.

	$v = 2$	$v = 3$	$v = 4$
$\tau(v)$	$-\frac{1}{3} \pm 0.04$	1 ± 0.03	1 ± 0.02
$\lambda(v)$	2.316 ± 0.002	3.950 ± 0.005	4.065 ± 0.005
$A(v)$	0.305 ± 0.004	0.355 ± 0.010	0.306 ± 0.006

4. Discussion

For the square lattice site animals we have shown that the connective constant (cell growth parameter) exists for restricted valence 3 and, assuming the plausible form (2.6) for the n -dependence of $a_n(v)$, we have shown rigorously that the exponent τ must change on going from $v = 2$ to $v = 3$. The numerical results presented here suggest that τ is unity for $v = 3$, as it is for $v = 4$. However, not unexpectedly, $\lambda(v)$ changes smoothly as v is varied from 2 to 4.

Acknowledgments

SGW and AJG wish to acknowledge financial support from the SRC in the form of Senior Visiting Fellowships. SGW is grateful to the University of Bristol and especially D H Everett for their kind hospitality, and AJG is similarly indebted to King's College, University of London and C Domb. We wish to thank M F Sykes for providing the lattice count for 18-edge polygons and J L Martin for allowing us to use his animal counting program.

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